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Range of Charges on Orthogonally Closed Subspaces of an Inner Product Space

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It is still an open question whether the complete lattice $F(S)$ of all orthogonally closed subspaces of an incomplete inner product space S admits a nonzero charge. A negative answer would result in a new way of completeness characterization of inner product spaces. Many partial results have been established regarding what has now turned to be a highly nontrivial problem. Recently, in Dvurečenskij and Ṕtak (*Letters in Mathematical Physics*, **62**, 63–70, 2002) the range of a finitely additive state s on $F(S)$, $\dim S = \infty$, was shown to include the whole interval [0, 1]. This was then generalized in Dvurečenskij (*International Journal of Theoretical Physics*, 2003) for general inner product spaces satisfying the Gleason property. Motivated by these results, we give a thorough investigation of the possible ranges of charges on $F(S)$, $\dim S \geq 3$. We show that if the nonzero charge m is bounded, then for infinite dimensional inner product spaces, Range(m) is always convex. We also show that this need not be the case with unbounded charges. Finally, in the last section, we investigate the range of charges on $F(S)$, $\dim S = \infty$, satisfying the sign-preserving and Jauch–Piron properties. We show that for such measures the range is again always convex.

KEY WORDS: Hilbert space; inner product space; orthogonally closed subspaces; completely additive signed measures; charges; sign-preserving property; Jauch–Piron property.

1. INTRODUCTION

Let H be a separable Hilbert space, $\dim H \geq 3$, and let $L(H)$ be the complete quantum logic consisting of closed subspaces of H. The cornerstone of quantum logic theory on $L(H)$ is Gleason's theorem (Dvurečenskij, 1992; Gleason, 1957; Ṕtak and Pulmannová, 1991; Varadajan, 1985), which asserts that for every state s on $L(H)$ there exists a unique Hermitian trace operator T on H with unit trace, such that

$$s(M) = \mathrm{tr}(TP_M), \qquad M \in L(H),$$

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where P_M denotes the orthoprojection of H onto M. This fundamental and highly nontrivial result is of crucial importance for the probabilistic theory of $L(H)$ and has many generalizations and applications (see, for example, Dvurečenskij, 1992). One of the fundamental consequences of Gleason's theorem related to the hidden variable hypothesis in quantum mechanics is the nonexistence of two-valued states on $L(H)$ where H is any Hilbert space of dimension at least 3 (see Alda, 1980; Chetcuti, 2002; Dvurečenskij, 1992).

In what follows, let S be an inner product space (real, complex, or quaternion) and let $\langle\cdot,\cdot\rangle$ be the inner product on S. For each subset $A \subset S$ let $A^{\perp} = \{b \in S : \langle a, b\rangle = 0 \text{ for all } a \in A\}$. Moreover for any two subspaces $A \subset B$ of S, we denote $A^{\perp} \cap B$ by $A^{\perp_B}$. Denoted by $F(S)$ the set of all *orthogonally closed subspaces of* S, that is

$$F(S) = \{M \subset S : M^{\perp\perp} = M\},$$

and let $P_0(S)$ be the family of all the finite dimensional subspaces of S. Then (see, for example Maeda and Maeda, 1970), $F(S)$ is a complete lattice, where for the meet we have

$$\bigwedge_{i\in I} M_i = \bigcap_{i\in I} M_i,$$

while for the join we have

$$\bigvee_{i\in I} M_i = \left(\mathrm{sp}\left(\bigcup_{i\in I} M_i\right)\right)^{\perp\perp}.$$

We recall that $F(S)$ does not have to be orthomodular,[3] in fact Amemiya and Araki (1966) proved the following algebraic criterion for the (topological) completeness of an inner product space S; an inner product space S is complete if and only if $F(S)$ is orthomodular. In the sequel, unless otherwise stated, we shall not assume the completeness of S.

A *completely additive signed measure* on $F(S)$ is a mapping $m : F(S) \to \mathbb{R}$ such that, if $\{M_i : i \in I\}$ is any collection of orthogonal elements in $F(S)$ then

$$m\left(\bigvee_{i\in I} M_i\right) = \sum_{i\in I} m(M_i). \tag{1.1}$$

A *charge* m on $F(S)$ is a mapping $m : F(S) \to \mathbb{R}$ such that eq. (1.1) holds only for finite orthogonal collections. A state s is a σ-additive measure such that its range lies in the interval [0, 1] and $s(S) = 1$. (A finitely additive state is defined analogously.)

Since Hilbert spaces form a proper subfamily of the class of inner product spaces (over the same field), it is indispensable to have completeness criteria

[3] $F(S)$ is said to be orthomodular if for every $A, B \in F(S)$, $A \subset B$, we have $B = A \vee (A^{\perp} \wedge B)$.

that characterize the complete among the incomplete spaces. The first measure-theoretic completeness criterion for inner product spaces was presented by Hamhalter and Pták in 1986 (Homhalter and Ṕtak, 1987). It was proved that for a separable, real, incomplete inner product space S, $F(S)$ admits no σ-additive states. This result was then generalized and many new measure-theoretic completeness criteria were derived (Dvurečenskij, 1992). However, it is still unknown whether $F(S)$—for an incomplete inner product space S—admits any finitely additive states. We have only partial results concerning this problem (see Chetcuti and Dvurečenskij, 2003; Dvurečenskij *et al.*, 1990). Recently, it has been proved in Dvurečenskij and Ṕtak (2002) that if S is any inner product space of infinite dimension and s is any finitely additive state on $F(S)$ then Range$(s) = [0, 1]$. (This was then extended in Dvurečenskij (in press) to all generalized inner product spaces satisfying the "Gleason Property.")

Motivated by these results, we study the possible ranges of charges on $F(S)$. We shall see that when the charge is bounded, its range is always convex. We shall also show that the situation can be very different for unbounded charges. Finally, sign-preserving charges on $F(S)$, $\dim S = \infty$, are considered and we prove that for such a measure satisfying the Jauch–Piron property the range is also a convex set.

2. CHARGES ON $F(S)$ WHEN $S = H_n$ ($n \leq 3$)

In this section we shall investigate the possible range of a charge m on $L(H_n)$ where n is at least 3. If we let $\kappa = m(H_n)$, and denote by $[x]$ the one-dimensional subspace in H_n spanned by x, we show that if m is bounded, then we either have $m([x]) = \frac{\kappa}{n}$ for all $x \in H_n$ (and therefore Range $(m) = \{0, \frac{\kappa}{n}, \frac{2\kappa}{n}, \ldots, \kappa\}$), or else there exist $a, b \in \mathbb{R}$, $a < b$, such that $[a, b] \subset$ Range(m).

The following is Gleason's theorem for bounded charges on $L(H_n)$. (See, for example Dvurečenskij (1992), Theorem 3.2.16.)

Theorem 2.1. *For every bounded charge m on $L(H_n)$, $n \geq 3$, there is a unique Hermitian trace operator T on H_n such that*

$$m(M) = \mathrm{tr}(TP_M), \qquad M \in L(H_n).$$

The following is a standard result in Functional Analysis. The reader may need to look at any book on Functional Analysis, for example, Kreyszig (1986).

Proposition 2.2. *Let H_n be an n-dimensional Hilbert space. Let T be a Hermitian trace operator on H_n. If $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$, are the eigenvalues of T corresponding to eigenvectors $x_1, x_2, \ldots, x_n$, then we have*

$$\lambda_1 = \max_{x \neq 0} \langle Tx, x \rangle \quad \textit{and} \quad \lambda_n = \min_{x \neq 0} \langle Tx, x \rangle.$$

Moreover

$$\lambda_i = \max_{0 \neq x \in Y_i} \langle Tx, x \rangle, \quad where \quad Y_i = \mathrm{sp}\{x_1, \ldots, x_{x-1}\}^{\perp}.$$

Although a proper extension, the following result is analogous to that in Dvurečenskij and, Ptak (2002).

Theorem 2.3. *Let m be a nonzero bounded charge on $L(H_n)(n \geq 3)$ and let $0 \leq \kappa = m(H_n)$. We either have that $m([x]) = \frac{\kappa}{n}$ for all $x \in H_n$ (and therefore* Range *$(m) = \{\frac{\kappa}{n}, \frac{2\kappa}{n}, \ldots, \kappa\}$), or else there exist $a, b \in \mathbb{R}$, $a < b$, such that $[a, b] \subset$* Range *(m). In addition,* $[\min_{x \neq 0} m([x]), \max_{x \neq 0} m([x])] \subset \mathrm{Range}(m)$.

Proof: It is clear that if we define $m : L(H_n) \to [0, \kappa]$ by $m([x]) = \frac{\kappa}{n}$ for all $x \in H_n$, then m is a positive charge on $L(H_n)$. We shall call this charge the trivial discrete charge on $L(H_n)$. Suppose that this is not our measure, that is, let $\lambda < \mu$ such that $m([x]) = \lambda$ and $m([y]) = \mu$ where $x, y \in H_n$. By Theorem 2.1, there exists a unique Hermitian trace operator T on H_n such that $m(M) = \mathrm{tr}(TP_M)$ for any $M \in L(H_n)$. The spectral theory of trace Hermitian operators yields that

$$m(M) = \sum_{i \leq n} \lambda_i \langle P_M x_i, x_i \rangle, \qquad M \in L(H_n),$$

where $\lambda_i = m([x_i]) = \langle Tx_i, x_i \rangle$ is the eigenvalue of T corresponding to the proper eigenvector x_i. It is evident that $\sum_{i \leq n} m([x_i]) = \kappa$ and therefore if $m([x_i]) = m([x_j])$ for all $i < j \leq n$, then m is the trivial discrete measure defined above. So we can pick x_{i_1}, x_{i_2} from $\{x_i : i \leq n\}$ such that $\lambda_{i_1} = m([x_{i_1}]) < m([x_{i_2}]) = \lambda_{i_2}$. Let

$$y = x_{i_1} \cos\phi + x_{i_2} \sin\phi, \qquad \phi \in \left[0, \frac{\pi}{2}\right].$$

Then

$$\begin{aligned} m([y]) &= \sum_{i \leq n} \lambda_i |\langle y, x_i \rangle|^2 \\ &= \lambda_{i_1} \cos^2\phi + \lambda_{i_2} \sin^2\phi \\ &= a \cos^2\phi + b \sin^2\phi, \end{aligned}$$

where we have identified λ_{i_1} and λ_{i_2} with a and b respectively. It is clear that we have $[a, b] \subset \mathrm{Range}(m)$. □

Proposition 2.4. *Let m be a nonzero bounded charge on $L(H_n)$, $(n \geq 3)$, not equal to the "trivial discrete charge." If $a = m(A) < m(B) = b$, where $A, B \in L(H_n)$, $A \perp B$,* $\dim A = \dim B$, *then* $[a, b] \subset \mathrm{Range}(m)$.

Proof: By Gleason's Theorem 2.1. there exists an orthonormal set of vectors $\{x_i : i \le n\}$ and a set $\{\lambda_i : i \le n\}$ of real numbers such that

$$m(M) = \sum_{i \le n} \lambda_i \langle P_M x_i, x_i \rangle, \qquad M \in L(H_n).$$

Let $A = [a_1] \oplus [a_2] \oplus \cdots \oplus [a_p]$ and $B = [b_1] \oplus [b_2] \oplus \cdots \oplus [b_p]$ be two orthogonal subspaces such that $a = m(A) < m(B) = b$. (We take $\{a_q : q \le p\}$ and $\{b_q : q \le p\}$ to be normalized.) For every $q \le p$, we define

$$y_q = a_q \cos\phi + b_q \sin\phi, \qquad \phi \in \left[0, \frac{\pi}{2}\right],$$

and put $Y = [y_1] \oplus [y_2] \oplus \cdots \oplus [y_p]$. Then

$$\begin{aligned} m(Y) &= \sum_{i \le n} \lambda_i \langle P_Y x_i, x_i \rangle \\ &= \sum_{i \le n} \lambda_i \sum_{q \le p} |\langle y_q, x_i \rangle|^2 \\ &= \sum_{q \le p} \sum_{i \le n} \lambda_i |\langle a_q, x_i \rangle \cos\phi + \langle b_q, x_i \rangle \sin\phi|^2 \\ &= m(A)\cos^2\phi + m(B)\sin^2\phi \\ &\quad + \cos\phi \sin\phi \sum_{q \le p} \sum_{i \le n} \lambda_i \{\langle a_q, x_i \rangle \langle x_i, b_q \rangle + \langle b_q, x_i \rangle \langle x_i, a_q \rangle\} \\ &= m(A)\cos^2\phi + m(B)\sin^2\phi + \gamma \cos\phi \sin\phi, \end{aligned}$$

where $\gamma \in \mathbb{R}$. By elementary real analysis theory, it follows that $[a, b] \subset \mathrm{Range}(m)$. This completes the proof. □

We now show that the condition "bounded" in Theorem 2.3 and Proposition 2.4 is not superfluous. Indeed we exhibit an unbounded charge m_0 on $L(H_n)$ such that no interval in $\mathbb{R}$ is included in its range.

Proposition 2.5. *For any $n \ge 2$, there exists an unbounded charge m_0 on $L(H_n)$ such that* Range(m_0) *is countable and infinite.*

Proof: Let $n = 2$. Take a sequence of one-dimensional subspaces $\{M_n : n \in \mathbb{N}\}$ which contains no orthogonal pairs. Define a mapping $m_0 : L(H_2) \to \mathbb{R}$ via

$$m_0(M) = \begin{cases} 0 & \text{if } M = \{0\}, \\ 1 & \text{if } M = H_2, \\ n+1 & \text{if } M = M_n, \\ -n & \text{if } M = M_n^{\perp}, \end{cases}$$

and on other one-dimensional subspaces M, $M^\perp$ choose $m_0(M) \in \{2, -1\}$ such that $m_0(M) + m_0(M^\perp) = 1$. Then m_0 is an unbounded charge and Range$(m_0) = \mathbb{Z}$.

For the case when $n \geq 3$, we make use of the result of Hamel (1905). This states that there exists a discontinuous additive functional $\phi : \mathbb{R} \to \mathbb{R}$. He proved that there exists a subset $V = \{s_i : i \in I\} \subset \mathbb{R}$ such that every real number can be uniquely represented as $r = \Sigma_{i \in I_0} \beta_i s_i$ where I_0 is a finite subset of I, $s_i \in V$ and $\beta_i \in \mathbb{Q}$. It is clear that V contains at least one irrational number s_1. (In fact, V includes at most one rational number.) If we put $\phi(\Sigma_{i \in I_0} \beta_i s_i) = \beta_1$, then ϕ is the functional in question. For any $x \in H_n$, $\|x\| = 1$, define the map $m_0 : L(H_n) \to \mathbb{R}$ by $m_0 = \phi \circ s_x$ where s_x is the Gleason state on $L(H_n)$ defined by $s_x(M) = \langle P_M x, x \rangle$, $M \in L(H_n)$. It can be seen that m_0 is a charge on $L(H_n)$ and since Range$(m_0) \subset \mathbb{Q}$, it follows that Range(m_0) is countable. Indeed, it can be proved (see Dvurečenskij, 1992, Proposition 3.2.4) that m_0 is unbounded. □

3. BOUNDED CHARGES ON $F(S)$ WHEN dim $S = \infty$

Here we show that for any nonzero bounded charge m on $F(S)$, $\dim S = \infty$, its range, Range(m) is a convex set.

In what follows let m be a bounded charge on $F(S)$, $\dim S = \infty$. Let $0 \leq k = m(S)$ and define $\lambda, \mu \in \mathbb{R}$ as follows:

$$\lambda = \inf\{m(A) : A \in F(S)\},$$
$$\mu = \sup\{m(A) : A \in F(S)\}.$$

It can be immediately noted that $\lambda + \mu = \kappa$. We prove that Range(m) contains the interval (λ, μ).

We shall need the following lemma. This is a generalization of the technique of Alda (1980). With this technique we embed $L(H_n)(n \geq 1)$ into $F(S)(\dim S = \infty)$. This was originally used to show that for any inner product space S, $F(S)$ admits no two-valued finitely additive states. (See, for example, Chetcuti, 2002; Dvurečenskij, 1992.) We remark that as in this case, this lemma is the state of art in Dvurečenskij (in press) and Dvurečenskij and Ṕtak (2002). We give a sketch of the proof for completeness.

Lemma 3.1. *Let S be an inner product space,* $\dim S = \infty$. *For any $n \geq 1$ there exists a mapping $\phi : L(H_n) \to F(S)$ such that for all $M, N \in L(H_n)$,*

(i) $\phi(H_n) = S$,
(ii) $\phi(M^\perp) = \phi(M)^\perp$,
(iii) *if $M \subset N$, then $\phi(M \vee N) = \phi(M) \vee \phi(N)$.*

Proof: Let $n \geq 1$ be given and let $\{x_i : i \in I\}$ be a maximal orthonormal system in S. We partition I in n disjoint subsets $\{I_k : k \leq n\}$ such that $|I| = |I_k|$ for all $k \leq n$. So I can be expressed in the form of disjoint union of n-element sets I_α. Put $H_\alpha = sp\{x_i : i \in I_\alpha\}$. For every α, let $T_\alpha : H_n \to H_\alpha$ be a unitary operator (i.e., T_α is bijective and $\langle T_\alpha x, T_\alpha y\rangle = \langle x, y\rangle$ for any $x, y \in H_n$). Define the map $\phi : L(H_n) \to F(S)$ by $\phi(M) = \vee_\alpha(T_\alpha M)$. It is not a difficult task to check that ϕ satisfies (i), (ii), and (iii) of Lemma 3.1. $\square$

Corollary 3.2. (1) *If m is a charge on $F(S)$ then $m \circ \phi$ defines a charge on $L(H_n)$.* (2) *For any inner product space S,* $\dim S \geq 3$, *$F(S)$ admits no two-valued charges.*

Lemma 3.3. *Let m be a charge on $F(S)$. Let $A \in F(S)$ and let $\{a_i : i \in I\}$ be a maximal orthonormal system*[4] *in A. Then $m(A) = m(\bigvee_{i\in I}[a_i])$.*

Proof: We certainly have that

$$(\mathrm{sp}\{a_i : i \in I\})^{\perp\perp} = \bigvee_{i\in I}[a_i] \subset A.$$

It is not difficult to verify that

$$\begin{aligned} m(A) &= m\left(\bigvee_{i\in I}[a_i]\right) + m((\mathrm{sp}\{a_i : i \in I\})^{\perp} \cap A) \\ &= m\left(\bigvee_{i\in I}[a_i]\right). \end{aligned}$$

$\square$

For any $A \in F(S)$, define

$$F(0, A) = \{B \in F(S) : B \subset A\},$$
$$F(A) = \{B \subset A : B^{\perp_A\perp_A} = B\}.$$

Observe that $F(0, A)$ and $F(A)$ are complete lattices with A and $\{0\}$ being the largest and smallest elements respectively. In the following, let us agree to denote by $\vee$, $\vee_{F(0,A)}$, and $\vee_{F(A)}$ the joins taken in $F(S)$, $F(0, A)$, and $F(A)$ respectively.

Proposition 3.4. *Let S be any inner product space and $A \in F(S)$. We have*

(i) $F(A) \subset F(0, A)$.
(ii) *If $F(E) = F(0, E)$ for all $E \in F(A)$, then A is complete.*
(iii) *Let m be a charge on $F(S)$. If $\{B_i : i \in I\}$ is any system of subspaces in $F(A)$, then $m(\bigvee^{F(A)}_{i\in I} B_i) = m(\bigvee_{i\in I} B_i)$. In particular, the restriction of m on $F(A)$ defines a charge.*

[4] denoted by MONS in short.

Proof: For the proof of (i) and (ii), the reader is referred to Chetcuti and Dvurečenskij (submitted). We prove statement (iii). Let $\{B_i : i \in I\} \subset F(A)$. Since $F(A) \subset F(0, A)$, we have that $\bigvee_{i\in\mathbb{N}} B_i \subset \bigvee^{F(A)}_{i\in\mathbb{N}} B_i$. Moreover,

$$\bigvee^{F(A)}_{i\in\mathbb{N}} B_i = \left(\bigvee_{i\in\mathbb{N}} B_i\right)^{\perp_A\perp_A},$$

and therefore,

$$m\left(\bigvee^{F(A)}_{i\in\mathbb{N}} B_i\right) = m\left(\bigvee_{i\in\mathbb{N}} B_i\right).$$

From this it is clear that the restriction of every charge m on $F(A)$ defines a charge. □

Lemma 3.5. *Let m be a bounded charge on $F(S)$, $\dim S = \infty$, and let $m(s) = \kappa \geq 0$. Then, the interval $[0, \kappa] \subset \mathrm{Range}(m)$.*

Proof: Let $M \in F(S)$ such that $\dim M = \dim M^\perp = \dim S$. Without any loss of generality we may assume that $m(M) \geq m(M^\perp)$. Then $m(M) > \frac{\kappa}{2}$. Let $n_0 = \max\{2, \frac{\kappa}{m(M)}\}$ and let $\{x_i : i \in I\}$ be a MONS in M. Then by Lemma 3.3, $m(M) = m(\bigvee_{i\in I}[x_i])$. Let $\{y_i : i \in I\}$ be a MONS in $M^\perp$. We can partition I into $n_0 - 1$ disjoint subsets $\{I_k : k \leq n_0 - 1\}$ such that $[I] = |I_k|$, $k \leq n_0 - 1$.

Using the usual technique, we can therefore express the set I in the form of a disjoint union of $(n_0 - 1)$-element sets I_α. Moreover, there is a one-to-one correspondence between these I_α's and I. So we can replace the indexing i in $\{x_i : i \in I\}$ with α. Put $H_\alpha = \mathrm{sp}\{x_\alpha \cup \{y_i : i \in I_\alpha\}\}$. Fix $y \in H_{n_0}$, $\|y\| = 1$ and let $T_\alpha : H_{n_0} \to H_\alpha$ be a unitary operator such that $T_\alpha y = x_\alpha$. Define $\phi : L(H_{n_0}) \to F(S)$ by $\phi(M) = \bigvee_\alpha (T_\alpha M)$ as in Lemma 3.1. It is clear that $\phi([y]) = \bigvee_{i\in I}[x_i]$. Since m is bounded, the state $m \circ \phi$ induced on $L(H_{n_0})$ is also bounded. In addition, $m \circ \phi([y]) = m(M)$. There exists $u \in H_{n_0}$ such that $m \circ \phi([u]) \leq \frac{\kappa}{n_0} < m(M)$ and therefore by Proposition 2.3 it follows that $[\frac{\kappa}{n_0}, m(M)] \subset \mathrm{Range}(m)$. Then $\cup_{n>n_0}[\frac{\kappa}{n}, m(M)] \subset \mathrm{Range}(m)$, i.e., $[0, m(M)] \subset \mathrm{Range}(m)$. If $m(M) \geq \kappa$, then obviously we have $[0, \kappa] \subset \mathrm{Range}(m)$. If $m(M) < \kappa$, then by considering complements, we conclude that $[0, k] \subset \mathrm{Range}(m)$. □

As a corollary we have the original result by Dvurečenskij and Pták (2002).

Corollary 3.6. *Let s be a finitely additive state on $F(S)$, $\dim S = \infty$. Then* $\mathrm{Range}(s) = [0, 1]$.

Lemma 3.7. *Let m be a bounded charge on $F(S)$, $\dim S = \infty$, and let $m(S) = \kappa \geq 0$. For any $A \in F(S)$, we have either $[0, m(A)] \subset \mathrm{Range}(m)$, or $[m(A), 0] \subset \mathrm{Range}(m)$, (depending on the sign of $m(A)$).*

Proof: When $\dim A = \infty$, result follows immediately from the fact that the restriction of m on $F(A)$ is a bounded charge and from Lemma 3.5.

So we assume that $\dim A < \infty$. If $\kappa = 0$, then, since $\dim A^{\perp} = \infty$, we have either $[m(A^{\perp}), 0] \subset \text{Range}(m)$ or $[0, m(A^{\perp})] \subset \text{Range}(m)$. Since $m(A) = -m(A^{\perp})$, we have $[0, m(A)] \subset \text{Range}(m)$ or $[m(A), 0] \subset \text{Range}(m)$. Let us now consider the case when $\kappa > 0$. There are three possibilities: (i) $0 \leq m(A) \leq \kappa$, (ii) $0 < \kappa < m(A)$, and (iii) $m(A) < 0 < \kappa$. Case (i) follows trivially since $[0, m(A)] \subset [0, k] \subset \text{Range}(m)$. For case (ii), we have $m(A^{\perp}) = \kappa - m(A) < 0$, so that $[m(A^{\perp}), 0] \subset \text{Range}(m)$, which yields $[\kappa, m(A)] \subset \text{Range}(m)$, and consequently.

$$[0, m(A)] = [0, \kappa] \cup [\kappa, m(A)] \subset \text{Range}(m)$$

In case (iii), we have $m(A^{\perp}) = \kappa - m(A) > 0$, so that $[0, m(A^{\perp})] \subset \text{Range}(m)$, which gives $[m(A), 0] \subset \text{Range}(m)$.

We now give the main result of this section. □

Theorem 3.8. *Let m be a bounded charge on $F(S)$, $\dim S = \infty$. Then* Range(m) *is a convex set containing (λ, μ), where $\lambda = \inf\{m(A) : A \in F(S)\}$ and $\mu = \sup\{m(A) : A \in F(S)\}$.*

We note that the authors do not know whether in Theorem 3.8, $\lambda, \mu \in \text{Range}(m)$.

In the following example we show that our assumption that m is bounded is in fact indispensable. Indeed, we show that not even the $P_0(S)$-boundedness[5] is sufficient to guarantee that the Range(m) is uncountable.

Proposition 3.9. *For every separable Hilbert space H, $\dim H = \infty$, there exists an unbounded charge m_0 on $L(H)$ which is $P_0(H)$-bounded and* $\text{Range}(m_0) \subset \mathbb{Q}$.

Proof: Let H, $\dim H = \infty$, be a separable Hilbert space. In Kalmbach (1986) it was shown that there is a state s on $L(H)$ which vanishes on all the finite dimensional subspaces of H. By considering the Hamel function ϕ described in Proposition 2.5, we define a charge $m_0 = \phi \circ s$ on $L(H)$ which vanishes on all finite dimensional subspaces of H. From the definition of ϕ it is clear that $\text{Range}(m_O) \subset \mathbb{Q}$. □

Corollary 3.10. *Let m be a non-zero charge on $F(S)$, $\dim S = \infty$. If $|\text{Range}(m)| \leq \aleph_0$, then m is unbounded.*

[5] We say that a charge m is $P_0(S)$-bounded when the set $\{|m(M)| : M \subset S, \dim M < \infty\}$ is bounded.

4. SIGN-PRESERVING CHARGES ON $F(S)$, $\dim S = \infty$, SATISFYING THE JAUCH–PIRON PROPERTY

A charge m on $F(S)$ is said to satisfy the *sign-preserving property* (or we say that m is a sign-preserving charge) if for any countable collection $\{N_i : i \in \mathbb{N}\}$ of orthogonal finite dimensional subspaces in $F(S)$ satisfying $m(N_i) > 0$, (respectively, $m(N_i) < 0$) for all $i \in \mathbb{N}$, it follows that $m(\bigvee_{i\in\mathbb{N}}) \leq 0$, (respectively $m(\bigvee_{i\in\mathbb{N}} N_i) \leq 0$). It is easy to verify that if m is sign-preserving and $m(N_i) > 0$ for all $i \in \mathbb{N}$, then

$$m\left(\bigvee_{i\in\mathbb{N}} N_i\right) \geq \sum_{i\in\mathbb{N}} m(N_i) > 0.$$

It is clear that every completely additive signed measure m on $F(S)$ and every positive (negative) charge satisfies the sign-preserving property. Let H be a separable infinite-dimensional Hilbert space and let m_1 and m_2 be two different finitely additive states on the orthomodular lattice $L(H)(= F(H))$ of closed subspaces of H such that they vanish on all the finite dimensional subspaces of H (see Kalmbach, 1986). Then $m = m_1 - m_2$ is a non-zero charge on $L(H)$ satisfying the sign-preserving property, and m is neither positive (negative) nor completely additive. Moreover, if we let ϕ denote the discontinuous additive functional defined in Proposition 2.5, then $\phi \circ m$ is an unbounded sign-preserving charge.

On the other hand, let H be a separable Hilbert space with an ONB $\{x_n : n \in \mathbb{N}\}$. Define the state on $L(H)$ by $m_1(M) = \sum_{n=1}^{\infty} \frac{1}{2^n}\langle P_m x_n, x_n\rangle$, $M \in L(H)$, and let m_2 be any finitely additive state on $L(H)$ vanishing on all the finite dimensional subspaces of H. Then $m = m_1 - m_2$ is a bounded charge on $L(H)$ not satisfying the sign-preserving property. Indeed, let $M = \bigvee_{n\geq 2}[x_n]$. Then for any $n \geq 2$, $m([x_n]) = \frac{1}{2^n}$ but $m(M) = \frac{1}{2} - 1 = -\frac{1}{2}$.

A charge m on $F(S)$ is said to satisfy the *Jauch-Piron* property if for any collection $\{x_i : i \in I\}$ of pairwise orthogonal vectors satisfying $m([x_i]) = 0$, we have $m(\bigvee_{i\in I}[x_i]) = 0$.

In the previous section we have characterized the range of bounded charges on $F(S)$, $\dim S = \infty$. Although no such characterization seems to be possible for unbounded charges (as hinted out in Proposition 3.9), we shall show that the range of sign-preserving charges satisfying the Jauch-Piron property is always convex.

We say that an element $A \in F(S)$ is *positive* (*negative*) *with respect* to a charge m if $m(N) \geq 0\,(m(N) \leq 0)$ for any $N \in F(0, A)$.

Lemma 4.1. *Let m be a sign-preserving charge on $F(S)$.*

(i) *If $\{x_i : i \in I\}$ is a system of pairwise orthogonal unit vectors of S, then the set $I_0 = \{i \in I : m([x_i]) \neq 0\}$ is countable.*

(ii) *If, in addition, m satisfies the Jauch-Piron property and $A \in F(S)$ such that $m([x]) \leq 0$ for all $x \in A$, then m is negative on A.*

Proof:

(i) Let $N_1 = \{i \in I : m([x_i]) < -1\}$. For every $k \in \mathbb{N}$, $k \geq 2$, let

$$N_k = \left\{ i \in I : -\frac{1}{k-1} \leq m([x_i]) < -\frac{1}{k} \right\}.$$

Similarly, let $P_1 = \{i \in I : m([x_i]) > 1\}$, and

$$P_k = \left\{ i \in I : \frac{1}{k-1} \geq m([x_i]) > \frac{1}{k} \right\}.$$

It is not difficult to check that because of the sign-preserving property of m, $|N_k|, |P_k| < \infty$ for all $k \in \mathbb{N}$. This implies that $I_0 = \bigcup_{k \in \mathbb{N}} N_k \cup \bigcup_{k \in \mathbb{N}} P_k$ is countable.

(ii) Let $N \in F(0, A)$ and let $\{x_i : i \in I\}$ be a MONS in N. By (i) above, the set $I' = \{i \in I : m([x_i]) < 0\}$ is countable. Then we have

$$m(N) = m\left(\bigvee_{i \in I} [x_i] \right) = m\left(\bigvee_{i \in I'} [x_i] \right) + m\left(\bigvee_{i \in I \setminus I'} [x_i] \right) < 0,$$

by the Jauch-Piron and sign-preserving properties.

□

Lemma 4.2. *Let m be a sign-preserving charge on $F(S)$, $\dim A = \infty$ such that $m(A) < 0$. For every $\epsilon > 0$, there exists $E \in F(0, A)$ such that*

(i) $E^{\perp_A} \in P_0(A)$;
(ii) $m(E) \leq m(A)$;
(iii) $m(B) \leq \epsilon$ *for all* $B \subset P_0(E)$.

Proof: The proof of this lemma resembles the proof of the Hahn-decomposition theorem (1988). (Here we do not have the σ-additivity of m, however, we shall see that with some "weak" form of additivity it is still possible to "do the trick.") Let $\epsilon > 0$ be given. If $m(B) \leq \epsilon$ for all $B \in P_0(A)$, then we put $E = A$ and we are done. Otherwise, let $B_1 \in P_0(A)$ such that $m(B_1) > \epsilon$. Put $A_1 = B_1^{\perp_A} \in F(A) \subset F(0, A)$. We note that $m(A_1) = m(A) - m(B_1) < m(A) - \epsilon < m(A)$. If $m(B) \leq \epsilon$ for all $B \in P_0(A_1)$ then A_1 is our subspace. If not, there exists $B_2 \in P_0(A_1)$ satisfying $m(B_2) > \epsilon$. Let $A_2 = B_2^{\perp_{A_1}} \in F(A_1) \subset F(0, A)$. It is easy to check that $m(A_2) < m(A)$ and therefore $A_2 \neq \{0\}$. Continuing like this, we can either after i steps arrive at some infinite dimensional subspace $A_i = B_i^{\perp_{A_{i-1}}}$ satisfying $m(A_i) < m(A) - i\epsilon$ and $m(B) \leq \epsilon$ for all $B \in P_0(A_i)$, or else, for contradiction, we define a nested sequence of subspaces $A = A_0 \supset A_1 \supset A_2 \supset \cdots$, where $A_i \neq A_j$ for $i > j$. We also have the sequence $\{B_i : i \in \mathbb{N}\}$ of orthogonal subspaces in $P_0(A)$

such that $m(B_i) > \epsilon$ and $A^{\perp_A} = B_1 \vee_{F(A)} \ldots \vee_{F(A)} B_i$. Then we have

$$\begin{aligned}
m\left(\bigcap_{i\in\mathbb{N}} A_i\right) &= m(A) - m\left(\bigvee_{\substack{F(A)\\ i\in\mathbb{N}}} A_i^{\perp_A}\right)\\
&= m(A) - m\left(\bigvee_{\substack{F(A)\\ i\in\mathbb{N}}} B_i\right)\\
&= m(A) - m\left(\bigvee_{\substack{F(A)\\ i\leq\mathbb{N}}} B_i\right) - m\left(\bigvee_{\substack{F(A)\\ i>\mathbb{N}}} B_i\right)\\
&< m(A) - n\epsilon \qquad \text{for all } n \in \mathbb{N},
\end{aligned}$$

which is the required contradiction. □

Proposition 4.3. *Let m be a sign-preserving charge on $F(S)$ satisfying the Jauch-Piron property. If $A \in F(S)$ is a subspace such that $m(A) < 0$, then there exists $M \in F(0, A)$, which is negative with respect to m, and $m(M) \leq m(A)$.*

Proof: According to Proposition 4.1, there is a finite dimensional subspace N of A such that $m(N) < 0$.

Either A is negative with respect to m or it contains a finite dimensional subspace of positive measure. In the second case, let n_1 be the smallest integer such that there exists a finite dimensional subspace $N_1 \subset A$ with $m(N_1) > \frac{1}{n_1}$. Either $N_1^{\perp_A}$ is of negative measure, or we repeat the first step.

Continuing successively, let n_k be the smallest integer for which there exists a finite dimensional subspace $N_k \subset (\bigvee_{i\leq k-1} N_i)^{\perp_A}$ such that $m(N_k) > \frac{1}{n_k}$. Now, we either finish after finitely many steps or find an infinite sequence of finite dimensional subspaces $\{N_i : i \in \mathbb{N}\}$. Let us put $M = (\bigvee_{i\in\mathbb{N}} N_i)^{\perp_A}$. For any integer $k \geq 1$, we have

$$\begin{aligned}
m(A) &= m(M) + m\left(\left(\bigvee_{i\in\mathbb{N}} N_i\right)^{\perp_A\perp_A}\right)\\
&= m(M) + m\left(\bigvee_{i\in\mathbb{N}} N_i\right)\\
&= m(M) + m\left(\bigvee_{i>k} N_i\right) + \sum_{i\leq k} m(N_i)\\
&> m(M) + \sum_{i\leq k} m(N_i) > m(N) + \sum_{i\leq k}\frac{1}{n_i}.
\end{aligned}$$

Therefore, the series $\sum_{i\in\mathbb{N}} \frac{1}{n_i} < \infty$, and $n_i \to \infty$. It is clear that $m(M) < m(A)$.

To show that A is negative with respect to m, choose $\epsilon > 0$. Since $n_i \to \infty$, we can find a sufficiently large integer i such that $\frac{1}{n_i - 1} < \epsilon$. Because of the inclusion $M \subset (\bigvee_{j \leq i} N_j)^{\perp_A}$, N cannot contain any finite dimensional subspace with measure greater than $\frac{1}{n_i - 1}$ which is smaller than ϵ. Consequently, N does not contain any finite dimensional subspaces with measure greater than ϵ. The arbitrariness of ϵ entails that N cannot contain any finite dimensional subspace of positive measure, and it has to be negative with respect to m. □

Lemma 4.4. *Let m be a sign-preserving charge on $F(S)$ satisfying the Jauch-Piron property. Let $A \in F(S)$, $\dim A = \infty$. Then we have either $[m(A), 0] \subset \mathrm{Range}(m)$, or $[0, m(A)] \subset \mathrm{Range}(m)$, (depending on the sign of $m(A)$).*

Proof: We can assume that $m(A) < 0$, since result is obviously true when $m(A) = 0$, and the case when $m(A) > 0$ would then follow by symmetry. By applying Lemma 4.2 on A with $\epsilon = 1$, we get $A = B_1 \oplus B_1^{\perp_A}$, where $\dim B_1 < \infty$ and $m(B) \leq 1$ for all $B \in P_0(B_1^{\perp_A})$. Let $A_1 = B_1^{\perp_A}$. Apply again the result of Lemma 4.2 on A_1 with $\epsilon = \frac{1}{2}$. Then $A_1 = B_2 \oplus B_2^{\perp_{A_1}}$. Continuing like this, we get a sequence of orthogonal finite dimensional subspaces $\{B_1, B_2, \ldots\}$ in A and a nested sequence $A \supset A_1 \supset A_2 \supset \cdots$ such that $m(B) \leq \frac{1}{k}$ for all $B \in P_0(A_k)$ and $m(A_k) \leq m(A)$. We have to consider two possibilities.

(i) There exists $N \in \mathbb{N}$ such that for all $n \geq N$, $B_n = 0$.
(ii) For all $N \in \mathbb{N}$ there exists $n > N$ such that $B_n \neq 0$.

Assume that we have case (i). This would imply that for all $B \in P_0(A_N)$, we have $m(B) \leq 0$ and therefore by Corollary 4.1, $m(B) \leq 0$ for any $B \in F(A_N)$. This implies that m is bounded on $F(A_N)$ and thus by Lemma 3.5, it follows that $[m(A_N), 0] \subset \mathrm{Range}(m)$. This implies that $[m(A), 0] \subset \mathrm{Range}(m)$.

In the other case, we have an infinite sequence $\{B_{n_i} : i \in \mathbb{N}\}$ of non-zero finite dimensional subspaces satisfying $m(B) \leq \frac{1}{n_i - 1}$ for all $B \in P_0(B_{n_i})$ and $m(B_{n_i}) > \frac{1}{n_i}$. We also have an infinite sequence of strictly nested subspaces $A \supsetneq A_{n_1} \supsetneq A_{n_2} \supsetneq \cdots$ such that $A_{n_{i-1}} = B_{n_i} \oplus A_{n_i}$. By nothing that

$$A_{n_i}^{\perp_A} = \bigvee_{\substack{F(A) \\ j \leq i}} B_{n_j},$$

it can be seen that

$$\begin{aligned} m\left(\bigcap_{i\in\mathbb{N}} A_{n_i}\right) &= m(A) - m\left(\bigvee_{\substack{F(A)\\ i\in\mathbb{N}}} B_{n_i}\right) \\ &= m(A) - m\left(\bigvee_{\substack{F(A)\\ i\le k}} B_{n_i}\right) - m\left(\bigvee_{\substack{F(A)\\ i>k}} B_{n_i}\right) \\ &< m(A) - \sum_{i\le k} \frac{1}{n_i} \qquad \text{for all } k\in\mathbb{N}. \end{aligned} \tag{4.1}$$

It is evident that $\sum_{i\in\mathbb{N}} \frac{1}{n_i} < \infty$. Moreover, $m(B) \le 0$ for all $B \in P_0(\bigcap_{i\in\mathbb{N}} A_{n_i})$. If $\dim \bigcap_{i\in\mathbb{N}} A_{n_i} = \infty$, then we would argue in the same manner as we did in case (i) above and conclude immediately that $[m(A), 0] \subset \text{Range}(m)$. So we assume that $\dim \bigcap_{i\in\mathbb{N}} A_{n_i} = k$. (Note that $\bigcap_{i\in\mathbb{N}} A_{n_i} \ne 0$ Let $\Delta = m(A) - m(\bigcap_{i\in\mathbb{N}} A_{n_i})$. By Eq. (4.1), $\Delta > 0$. Since $\sum_{i\in\mathbb{N}} \frac{1}{n_i} < \infty$, there exists $j_0 \in \mathbb{N}$ such that

$$\sum_{i=j_0-1}^{j_0+k-1} \frac{1}{n_i} < \frac{\Delta}{2}.$$

Moreover, it is not difficult to check that for all $i \in \{j_0, \ldots, j_0+k\}$, there exists $x_i \in B_{n_i}$ such that

$$0 < m([x_i]) \le \frac{1}{n_{i-1}}.$$

Let

$$H = [x_{j_0}] \oplus [x_{j_0+1}] \oplus \cdots \oplus [x_{j_0+k}] \oplus \left(\bigcap_{i\in\mathbb{N}} A_{n_i}\right).$$

The dimension of H is $2k+1 \le 3$ and

$$\begin{aligned} m(H) &\le \sum_{i=j_0}^{j_0+k} \frac{1}{n_{i-1}} + m\left(\bigcap_{i\in\mathbb{N}} A_{n_i}\right) \\ &< \frac{\Delta}{2} + m\left(\bigcap_{i\in\mathbb{N}} A_{n_i}\right) \\ &< m(A). \end{aligned}$$

In addition, since $H \subset A_{n_{j_0-1}}$, we have $m(B) < \frac{1}{n_{j_0}-1}$ for all $B \in L(H)$. Thus m restricted on $L(H)$ is a bounded charge. Moreover, by the choice of the x_i's it is clear that we do not have $m([x]) = \frac{m(H)}{2k+1}$ for all $x \in H$. Put $C = \bigcap_{i\in\mathbb{N}} A_{n_i}$ and $D = [x_{j_0+1}] \oplus \cdots \oplus [x_{j_0+k}]$. By Proposition 2.4, we have $[m(C), m(D)] \subset$

Range(m), which implies that $[m(A), 0] \subset$ Range(m). This completes the proof of Lemma 4.4. $\square$

Theorem 4.5. *Let m be a sign-preserving charge on $F(S)$ satisfying the Jauch-Piron property where* $\dim S = \infty$. *Then* Range(m) *is convex and contains the interval (λ, μ), where $\lambda, \mu \in \mathbb{R} \cup \{-\infty, +\infty\}$ are defined as* $\lambda = \inf\{m(A) : A \in F(S)\}$ *and* $\mu = \sup\{m(A) : A \in F(S)\}$.

Proof: Using Lemma 4.4, we can prove, in a similar way as in Lemma 3.7, that for any $A \in F(S)$, we either have $[m(A), 0] \subset$ Range(m), or $[0, m(A)] \subset$ Range(m), (depending on the sign of $m(A)$). $\square$

We conclude this paper with some remarks. We know that if $F(S)$ admits a completely additive signed measure, then S is topologically complete. We still do not know whether $F(S)$, for an incomplete S, admits a non-zero charge. In Chetcuti and Dvurečenskij (submitted), it has been proved that any Jordan charge on $F(S)$, $\dim S \geq 3$, can be uniquely expressed as a sum $m = m_1 + m_2$, where m_1 is a $P_0(S)$-regular mapping on $F(S) \to \mathbb{R}$ is $P_0(S)$-regular if for any $M \in F(S)$ and any $\epsilon > 0$, we can find a finite dimensional subspace N of M such that $|m(M) - m(N)| < \epsilon$.) The question *If $F(S)$ admits a non-zero charge, does it imply that S is complete?* has not been answered yet. Here we ask: *If $F(S)$ admits a non-zero sign-preserving charge satisfying the Jauch-Piron property, does it follow that S is complete?* It would be interesting to know whether this "mild property" of a charge is enough to guarantee the completeness of S.

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REFERENCES

Alda, V. (1980). On 0-1 measures for projectors. *Aplik. matem.* **25**, 373–374.

Amemiya, J. and Araki, H. (1966). A remark on Piron's paper. *Research for Mathematical Sciences Publications*. (Kyoto). Ser A2, Koyoto Univ., 423–427.

Chetcuti, E. (2002). *Completeness Criteria for Inner Product Spaces*, MSc Thesis, University of Malta.

Chetcuti, E. and Dvurečenskij, A. (2003). *A finitely additive state criterion for the completeness of inner product spaces*. *Letters in Mathematical Physics* **65** (in press).

Dvurečenskij, A. (1992). *Gleason's Theorem and Its Applications*, Kluwer, Dordrecht, Ister Science Press, Bratislava.

Dvurečenskij, A. (2003). States on subspaces of inner product spaces with the Gleason property, *International Journal of Theoretical Physics* **42**, 1393–1401.

Dvurečenskij, A. and Pták, P. (2002). On states on orthogonally closed subspaces of an inner product space, *Letters in Mathematical Physics* **62**, 63–70.

Dvurečenskij, A., Neubrunn, T., and Pulmannová, S. (1990). Finitely additive states and completeness of inner product spaces, *Foundation Physics* **20**, 1091–1102.

Gleason, A. M. (1957). Measures on the closed subspaces of a Hilbert space, *Journal of Mathematical Mechanics* **6**, 885–893.

Halmos, P. R. (1988). *Measure Theory*, Springer-Verlag, New York.

Hamel, G. (1905). Eine Basis alen Zahlen und die unstetigen Lösungen der Funktionalgleichung: $f(x+y) = f(x) + f(y)$, *Mathematical Analysis* **60**, 459–462.

Hamhalter, J. and Pták, P. (1987). A completeness criterion for inner product spaces, *Bulletin of London Mathematical Soceity* **19**, 259–263.

Kalmbach, G. (1986) *Measures and Hilbert Lattices*, World Science Publishing Company, Singapoore.

Kreyszig, E. (1986). *Introductory Functional Analysis with Applications*, Wiley Classics Library, Canada.

Maeda, F. and Maeda, S. (1970). *Theory of Symmetric Lattices*, Springer-Verlag, Berlin.

Pták, P. and Pulmannová, S. (1991). *Orthomodular Structures as Quantum Logics*, Kluwer, Dordrecht, The Netherlands.

Varadajan, V. S. (1985). *Geometry of Quantum Theory*, Springer-Verlag, New York Inc.